NON-AUTONOMOUS DYNAMICAL SYSTEMS AND THEIR APPLICATIONS

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Doctor Habilitat in Physical and Mathematical Sciences, university professor **David CHEBAN**
Laureate of the ASM Prize in the field of mathematics and informatics "Vladimir Andrunachievici" for the years 2019–2020
E-mail: david.ceban@usm.md; davidcheban@yahoo.com
ORCID: https://orcid.org/0000-0002-2309-3823
State University of Moldova

SISTEMELE DINAMICE NEAUTONOME ȘI APLICAȚIILE LOR

Rezumat. Articolul reprezintă o scurtă trecere în revistă a cercetărilor efectuate de autor în ultimii 10-15 ani privind sistemele dinamice neautonome și aplicațiile acestora. Sistemele dinamice neautonome constituie un nou domeniu ce contribuie la dezvoltarea rapidă a matematicii (teoria sistemelor dinamice). Mii de articole, inclusiv zeci de articole de sinteză și un șir de monografii despre sistemele dinamice neautonome au fost publicate în ultimele decenii, iar problematica respectivă a făcut cap de afiș la conferințele internaționale. Autorul a publicat trei monografii pe problema sistemelor dinamice neautonome. În acest articol este oferită o prezentare generală a rezultatelor obținute.

Cuvinte-cheie: soluții periodice, soluții cvasi-periodice, soluții aproape periodice Bohr/Levitan, soluții Bohr aproape automorfe, soluții recurente Birkhoff, soluții stabile Lagrange, soluții aproape recurente, soluții stabile Poisson, stabilitate Lyapunov, stabilitate asimptotică, atractori globali.

Summary. This article is devoted to a brief overview of the author's works over the past 10-15 years on non-autonomous dynamic systems and their applications. Non-autonomous dynamical systems are a new and rapidly developing field of mathematics (theory of dynamical systems). Thousands of articles, dozens of reviews and a number of monographs on non-autonomous dynamic systems and their applications have been published over the past 10-15 years. Special international conferences and scientific journals are dedicated to them. My results on non-autonomous dynamical systems and their applications are published in three monographs. In this article, we provide an overview of these results.

Keywords: Periodic solution, Quasi-periodic solutions, Bohr/Levitan almost periodic solutions, almost automorphic solutions, Birkhoff recurrent solutions, Lagrange stable solutions, Almost recurrent solutions, Poisson stable solutions, Lyapunov stability, Asymptotic stability, Global Attractors.

1. INTRODUCTION

In this article we provide a short overview of our work over the non-autonomous dynamic systems and their applications. Conditionally, I would single out the following areas of my research in this area:

(i) Global attractors of non-autonomous dynamical systems;

(ii) Nonlinear oscillations in non-autonomous dynamical systems. Namely, Poisson-stable (periodic, almost periodic, almost automorphic, almost recurrent, recurrent and Poisson-stable) motions of dynamical systems;

(iii) Lyapunov stability of non-autonomous dynamical systems.

In accordance with this, the further presentation will consist of three parts, in each of which we will give a brief description of our results in the areas highlighted above.

2. GLOBAL ATTRACTORS OF NON-AUTONOMOUS DYNAMICAL SYSTEMS

In the qualitative theory of differential equations non-local problems play an important role, especially in regard to questions of boundedness, periodicity, almost periodicity, Poisson stability, asymptotic behavior, dissipativity etc.

The present work takes a similar approach and is dedicated to the study of abstract non-autonomous dissipative dynamical systems and their application to differential equations.

In applications there often occur systems

$$u' = f(t, u), \tag{2.1}$$

which have every one of their solutions driven into fixed bounded domain and kept there under further increase of time, because of natural dissipation. Such systems are called dissipative ones. Solutions of dissipative systems are called limit (finally) bounded.

Dynamical systems occur in hydrodynamics studying turbulent phenomena, meteorology, oceanography, theory of oscillations, biology, radio engineering and other domains of sciences and engineering technics related to the study of asymptotic behavior. Lately the interest in dissipative systems increased even more because of intensive elaboration of strange attractors.

The study of the dissipative systems can be found in plenty of works, beginning from the classical works of N. Levinson. Among works on dissipative systems of ordinary diferential equations two directions can be made out.

The first one includes the works, which contain some conditions assuring the dissipativity of the system (2.1), some classes or concrete systems, representing theoretical or applied interest.

The second direction refers to works in which inner conditions of dissipative systems are studied, that is, conditions relating to the solutions` behavior character of the system when assuming its dissipativity, for different classes of differential equations.

We note that all works mentioned above (with rare exceptions) studied periodical or autonomous systems.

If the right-hand side f of the equation (2.1) is non-periodic, e.g. quasi-periodic (almost periodic by Bohr, recurrent in the sense of Birkhoff, almost periodic by Levitan, stable by Poisson) or depending on time in more complicated way, then the situation essentially complicates already in the class of almost periodic systems. It is caused at least by two reasons.

First, the definition of dissipativity in the non-autonomous case needs to be made more precise because Levinson's definition in the class of non-periodical systems divides on some non-equivalent notions and we need to choose one which allows us to develop a general theory which would contain as particular case the most essential results obtained for periodical dissipative systems.

Second, in the study of periodic dissipative systems an important role is played by discrete dynamical systems (cascades) generated by the degrees of the Poincar´e's transformation (mapping). For non-periodic systems there is no Poincar´e's transformation and, consequently, the approach created for research on periodical dissipative systems is not useful in the more general case. That is why to study non-periodical dissipative systems we need new ideas; that is, making a theory of non-autonomous dissipative dynamical systems demands making corresponding methods of research.

Our approach to the study of dissipative systems of differential equations consists of drawing to the study of non-autonomous dissipative systems ideas and methods developed in the theory of abstract dynamical systems. We select one class of dynamical systems (called in this work, dissipative), modelling the properties of dissipative differential equations. The selected class is systematically researched and then the general results obtained are applied to the study of dissipative systems of differential and some other classes of equations.

The idea of applying methods of the theory of dynamical systems to the study of non-autonomous differential equations is not new. It has been successfully applied to the resolution of different problems in the theory of linear and non-linear non-autonomous differential equations for more than thirty years. First this approach to non-autonomous differential equations was introduced in works of V. M. Millionshchikov, B.A. Shcherbakov, L.G. Deyseach and G.R. Sell, R.K. Miller, G. Seifert, G.R. Sell, later in works of V.V. Zhikov, I.U. Bronshtein, R. A. Johnson and many other authors. This approach consists of naturally linking with equation (2.1) a pair of dynamical systems and a homomorphism of the first onto the second. In one dynamical system is put the information about right hand side of equation (2.1) and in the other about the solutions of equation (2.1).

We note that there exists another approach offered in the works of V.I. Zubov and then developed in works of C.M. Dafermos, J.K. Hale, I. Hitoshi and many other authors. It consists of linking with every non-autonomous differential equation a two-parametric family of mappings (by terms of some authors – process).

The author adheres to the first approach, because, in his opinion, it is better adapted for resolving those problems which are studied in this work.

Our main results in this domain were published in the monograph [2]. This book consists of seventeen chapters.

In the first chapter for autonomous dynamical systems the different kinds of dissipativity are introduced and studied: point, compact, local, bounded and weak one. Criteria of point, compact and local dissipativity are given. It is shown that for dynamical systems in locally compact spaces all three types of dissipativity are equivalent. Examples are given showing that in the general case the notions of point, compact and local dissipativity are different. The notion of Levinson's center, which is an important characteristic of compact dissipative systems, is introduced. The solution of J.K. Hale's problem for locally bounded dynamical systems is given.

The second chapter is dedicated to non-autonomous dissipative dynamical systems. It is noted that in the general case Levinson's center of a non-autonomous dissipative dynamical system is not orbitally stable. The question of stability of Levinson's center of non-autonomous systems is studied. A simple geometric description ensuring its stability is given, as is a description of Levinson's center of non-autonomous systems satisfying the condition of uniform positive stability. There are pointed conditions of keeping the property of dissipativity under homo-morphisms. It is selected a class of dynamical systems which allow full description of Levinson's center's structure, called in this work systems with convergence. Some criteria of convergence in terms of Lyapunov's functions depending on two space variables are given. It is shown that for non-autonomous dissipative dynamical systems with finite-dimensional phase space all three types of dissipativity are equivalent. These series of conditions that are equivalent to dissipativity in finite-dimensional space are given. At last, it is proved that for linear systems dissipativity reduces to convergence. Also, there are given series of conditions equivalent to dissipativity of linear systems.

The third chapter deals mostly with a special class of non-autonomous dissipative dynamical systems called in the work C-analytic. It is proved that C-analytic dissipative dynamical system has the property of uniform positive stability on compact subsets. Full description of Levinson's center of these systems is given. One general construction allowing to connect with given non-autonomous dynamical system an autonomous dynamical system in space of continuous sections is provided. With the help of these constructions are studied quasi-periodic solutions of analytic systems with quasi-periodic coefficients. In conclusion conditions are given which guarantee the dissipativity of weakly nonlinear systems of differential equations, as is the condition which assures the existence of almost periodic solution of weakly nonlinear system with almost periodic coefficients in Levinson's center.

The fourth chapter is dedicated to a study of Levinson's center's structure with condition of hyperbolicity on closure of recurrent motion's set. There we establish some topological properties of Levinson's center of compact dissipative dynamical system. In particular, it is shown that Levinson's center is indecomposable if the phase space of dynamical systems is also indecomposable. It is proved that in connected and locally connected space Levinson's center of compact dissipative dynamical system both with continuous and discrete time is a connected set. There we establish some properties of a set of chain recurrent motions of dissipative system. A theorem is proved about the spectral decomposition of Levinson's center which is analogous to the known Smale's theorem. For one-dimensional dissipative dynamical systems, it was proved a theorem which precises the theorem about spectral decomposition of Levinson's center and, particularly, it was shown that Levinson's center of such systems contains a local maximal hyperbolic Markov set. In the end of the chapter an application of obtained results to periodic systems is given.

In the fifth chapter we develop the method of Lyapunov's functions for research of non-autonomous dissipative dynamical systems in finite-dimensional space. Criteria of dissipativity in terms of Lyapunov's functions, with the help of which we can get sufficient tests for dissipativity suitable in applications, are discussed. With the help of Lyapunov's functions there were proved a series of tests for dissipativity of multi-dimensional non-autonomous differential equations. On the basis developed for research of non-autonomous dissipative systems methods a criterion of asymptotic stability of zero section of non-autonomous systems has been obtained. In particular, it is proved the analog of known Barbashin-Krasovsky's theorem for non-autonomous dynamical systems. There are established some tests for convergence of systems of differential equations with the help of Lyapunov's functions depending on two space variables. There were proved tests for dissipativity and convergence of some systems of differential equations of the 2nd and 3rd order appearing in applications.

The sixth chapter is dedicated to some applications of general results obtained in previous chapters to difference equations, equations with impulses, functional-differential equations and evolutionary equations $x' + Ax = f$ with uniformly monotone operator A. In particular, tests for dissipativity and convergence of weakly nonlinear systems of difference equations and equations with impulses are given. The criterion of asymptotic stability of linear functional-differential equations is proved. Test for convergence of evolutionary equation $x' + Ax = f$ with uniformly monotone operator A it is established.

In the seventh chapter we systematically study the problem of upper semi-continuity of compact global attractors and compact pullback attractors of abstract non-autonomous dynamical systems under small perturbations. Several applications of our results are given for different classes of evolutionary equations.

The eighth chapter is devoted to the study of the relationship between the global attractor of the skew-product system and the pullback and forward attractors of the cocycle system. We also note that forward attractors are stronger than global attractors if we suppose a compact set of non-autonomous perturbations. An example is presented in which the cartesian product of the component subsets of a pullback attractor is not a global attractor of the skew-product flow. This set is, however, a maximal compact invariant subset of the skew-product flow. By a generalization of some stability results of V.I. Zubov it is asymptotically stable. Thus, a pullback attractor always generates a local attractor of the skew–product system, but this does not need to be a global attractor. If, however, the pullback attractor generates a global attractor in the skew-product flow and if, in addition, its component subsets depend lower semi-continuously on the parameter, then the pullback attractor is also a forward attractor. Several examples illustrating these results are presented in the final section.

In the ninth chapter we systematically study the global pullback attractors of C-analytic cocycles. For a large class of C-analytic cocycles we give the description of the structure of their pullback attractors. Particularly we prove that it is trivial, i.e. the fibers of these attractors contain only one point. Several applications of these results are given (ODEs, Caratheodory's equations with almost periodic coefficients, almost periodic ODEs with impulse).

The tenth chapter is dedicated to the investigation of the effect of time discretization on the pullback attractor of a non-autonomous ordinary differential equation for which the vector rely on a parameter that varies in time rather than depending directly on time itself. The parameter space is assumed to be compact so the skew product flow formalism as well as cocycle formalism also applies and the vector fields have a strong dissipative structure that implies the existence of a compact set that absorbs all compact sets under the resulting non-autonomous dynamics. The numerical scheme considered is a general 1-step scheme such as the Euler scheme with variable time-steps. Our main result is to show that the numerical scheme interpreted as a discrete time non-autonomous dynamical system, hence discrete time cocycle mapping and skew product flow on an extended parameter space, also possesses a cocycle attractor and that its component subsets converge upper semi-continuously to those of the cocycle attractor of the original system governed by the differential equation. We will also see that the corresponding skew product flow systems have global attractors with the cocycle attractor component sets as their cross-sectional sets in the original state space. Finally, we investigate the periodicity and almost periodicity of the discretized pullback attractor when the parameter dynamics in the ordinary differential equation is periodic or almost periodic and the pullback attractor consists of singleton valued component sets, i.e. the pullback attractor is a single trajectory.

In the eleventh chapter we study the non-autonomous Navier-Stokes equations. It is proved that such systems admit compact global attractors. This problem is formulated and solved in the terms of general non-autonomous dynamical systems. We give conditions of convergence of non-autonomous Navier-Stokes equations. A test of existence of almost periodic (quasi periodic, recurrent, pseudo recurrent) solutions of non-autonomous Navier-Stokes equations is given. We prove the global averaging principle for non-autonomous Navier-Stokes equations.

The twelfth chapter is devoted to the investigation of the global attractors of general V-monotone non-autonomous dynamical systems and their applications to different classes of differential equations (ODEs, ODEs with impulse, some class of evolution partial differential equations).

In the thirteenth chapter we study the linear almost periodic dynamical systems. The bounded solutions, relation between different types of stability and uniform exponential stability for those systems are studied. We give several applications the obtained results for ODEs, PDEs and functional-differential equations.

Chapter 14 is devoted to the study of quasi-linear triangular maps: chaos, almost periodic and recurrent solutions, integral manifolds, chaotic sets etc. This problem is formulated and solved in the framework of non-autonomous dynamical systems with discrete time. We prove that such systems admit an invariant continuous section (an invariant manifold). Then, we obtain the conditions of the existence of a compact global attractor and characterize its structure. We give a criterion for the existence of almost periodic and recurrent solutions of the quasi-linear triangular maps. Finally, we prove that quasi-linear maps with chaotic base admit a chaotic compact invariant set.

Chapter 15 is dedicated to the study of the problem of existence of compact global attractors for control systems (both with continuous and discrete time) and to the description of its structure.

The aim of the Chapter 16 is studying the problem of uniform asymptotic stability of the switched system

$$x' = f_{\nu(t)}(x) \ (x \in E^n), \tag{2.2}$$

where $\nu : R_+ \to \{1, 2, \dots, m\}$ is an arbitrary piecewise constant function, E^n is an n-dimensional Euclidian space, and $R_+ := [0, +\infty)$. In this Chapter the problem of uniform asymptotic stability of the discrete switched system

$$u(k+1) = f_{\nu(k)}(u(k)) \ (u \in E^n), \tag{2.3}$$

where $\nu : Z_+ \to \mathcal{P} := 1, 2, \dots, m\}$ is an arbitrary piecewise constant function and $Z_+ := \{0, 1, 2, \dots\}$, is also studying.

Chapter 17 is dedicated to the study of absolute asymptotic stability of differential/difference equations and inclusions. We establish the relation between linear inclusions and non-autonomous dynamical systems (cocycles). In the framework of general non-autonomous dynamical systems (both linear and non-linear) we study the problem of asymptotic stability and absolute asymptotic stability for discrete linear inclusions. We also study asymptotic stability of switched systems. We show that every switched system generates a non-autonomous dynamical system (cocycle). Using this fact, we apply the ideas and methods of the theory of non-autonomous dynamical systems to the study the problem of asymptotic stability of different classes of switched systems (linear systems, homogeneous systems, slow switched systems etc.).

3. NONLINEAR OSCILLATIONS IN NON-AUTONOMOUS DYNAMICAL SYSTEMS

One of the fundamental questions of the qualitative theory of non-autonomous differential/difference equations is the problem of almost periodicity, or more generally Poisson stability (in particular, Levitan almost periodicity, Bochner almost automorphy, almost recurrence in the sense of Bebutov, recurrence in the sense of Birkhoff and so on) of solutions.

The theory of almost periodic functions was mainly created and published by H. Bohr (In this relation see also the important results of P. Bohl and E. Esclangon). Bohr's theory was substantially extended by S. Bochner, H. Weyl, A. Besicovitch, J. Favard, J. von Neumann, V.V. Stepanov, N.N. Bogolyubov and others.

B. M. Levitan introduced a new class of functions (the so-called *N*-almost periodic or Levitan almost periodic functions) which includes any Bohr almost periodic function, but do not coincide with the latter. The foundation of this type of functions was created in the works of B.M. Levitan, B.Yu. Levin and V.A. Marchenko. A notion of almost automorphic function was introduced by S. Bochner whichalso is an extension of Bohr almost periodicity. Some substantial results about almost automorphic functions were obtained by W. Veech.

The different classes of Poisson stable functions (in particular, recurrent in the sense of G. Birkhoff, almost recurrent in the sense of M.V. Bebutov, pseudo recurrent and so on) have been introduced and studied by B.A. Shcherbakov.

The theory of Bohr/Levitan almost periodic, almost automorphic and Poisson stable functions is widely presented in the monographs of L. Amerio and G. Prouse, S. Bochner, C. Corduneanu, Toka Diagana, A. Fink, J. Favard, Y. Hino, T. Naito, N. Van Minh and J. Shin, B.M. Levitan, B.M. Levitan and V.V. Zhikov, A.A. Pankov, G. M. N'Guerekata, B.A. Shcherbakov, W. Shen and Y. Yi, T. Yoshizawa, S. Zaidman and others.

In the last 25-30 years the theory of Bohr/Levitan almost periodic, almost automorphic and Poisson stable differential/difference equations has been developed in connection with problems of differential/difference equations, stability theory, dynamical systems, and so on. The main achievements are related to the application of ideas and methods of dynamic systems in the study of the above-mentioned problems.

Our main results about nonlinear oscillations in non-autonomous dynamical systems are published in the monograph [3].

This monograph is dedicated to the abstract theory of non-autonomous dynamical systems, which is a new branch of the theory of dynamical systems.

In this monograph, I present the developments of the basic ideas and methods for non-autonomous dynamical systems and their applications over the past ten years.

Our main applications are non-autonomous ordinary differential/difference equations, functional differential/difference equations and some classes of partial differential equations.

In the recent years there seems to be a growing interest in non-autonomous differential/difference equations, both finite-dimensional (ordinary differential/difference equations) and infinite-dimensional (functional differential/difference equations and partial differential equations).

Nonlocal problems concerning the conditions of existence of different classes of solutions play an important role in the qualitative theory of differential equations. Here we include the problem of boundedness, periodicity, Bohr/Levitan almost periodicity, almost automorphy, almost recurrence in the sense of Bebutov, recurrence in the sense of Birkhoff, stability in the sense of Poisson, the problem of existence of limit regimes of different types, convergence, dissipativity etc.

The present work belongs to this direction and it is dedicated to the *mathematical theory of non-autonomous dynamical systems* and *applications*. The main goal of this book is to study Bohr/Levitan almost periodic, almost automorphic, different classes of Poisson stable motions and global attractors of Bohr/Levitan almost periodic systems with continuous and discrete time.

Thus, there are two key objects that are the subjects of study in this book. These are various *oscillatory regimes* (Bohr/Levitan almost periodic and Poisson stable movements) and *global attractors* and *application* of the obtained general results (related to abstract non-autonomous dynamical systems) to different classes of non-autonomous *differential* and *difference equations*.

The problems that we consider in this book are mainly motivated by non-autonomous differential/difference equations.

The monograph presents ideas and methods, developed by the author, to solve the problem of existence of Bohr/Levitan almost periodic (respectively, almost recurrent in the sense of Bebutov, almost authomorphic, Poisson stable) solutions and global attractors of non-autonomous differential/difference equations. Namely, the text provides answers to the following problems:

(i) Problem of existence of at least one Bohr/Levitan almost periodic solution for linear almost periodic differential/difference equations without Favard's separation condition (Favard theory);

(ii) Problem of existence of Bohr/Levitan almost periodic solution for monotone differential/difference equations;

(iii) Problem of existence of at least one Bohr/Levitan almost periodic solution for uniformly stable and dissipative differential equations (I.U. Bronshtein's conjecture, 1975);

(iv) Problem of description of the structure of the global attractor for holomorphic and gradient-like non-autonomous dynamical systems;

(v) Problem of existence of Levitan almost periodic solutions for linear differential equations (V.V. Zhikov's problem, 1971).

Plenty of work is dedicated to the study of problem of Bohr/Levitan almost periodicity, almost automorphy and different classes of Poisson stability of solutions for differential/difference equations. We survey briefly some of these works in our book.

Note that a bibliography of papers on Bohr/Levitan almost periodic, almost automorphic and Poisson stable solutions of almost periodic differential/difference equations contains over 300 items, i.e., it is still a very active area of research.

The body of the book consists of six chapters.

In the first chapter for semigroup dynamical systems there are introduced and studied different kinds of Poisson stability of motions and their comparability by character of recurrence: Bohr/Levitan almost periodicity, almost automorphy, Bebutov almost recurrence, Birkhoff recurrence, pseudo recurrence and other types of Poisson stability.

The second chapter is dedicated to the study of compact global attractors of dynamical systems (the both autonomous and non-autonomous).

The third chapter is dedicated to the study of holomorphic non-autonomous dissipative dynamical systems. We prove that a holomorphic dissipative dynamical system has the property of uniform positive stability on compact subsets. We study the holomorphic discrete dynamical systems on the infinite-dimensional spaces. The positive answer for Belitskii-Lyubich conjecture (for holomorphic discrete dynamical systems and flows) is given.

In the fourth chapter we present some new results about Bohr/Levitan almost periodic, almost automorphic and Poisson stable solutions of linear differential equations which complement the classical theory of Favard. In conclusion, we give conditions which guarantee the dissipativity of semi-linear systems of differential equations, as is, for example, the condition which assures the existence of almost periodic solutions of semi-linear system with almost periodic coefficients in Levinson's center.

The fifth chapter is dedicated to the study of order-preserving non-autonomous dynamical systems. We give some criteria of existence of a fixed point for a semi- group of transformations. The problem of existence of Bohr/Levitan almost periodic, almost automorphic and Poisson stable solutions for different classes of monotone differential equations (first order, second order, finite-dimensional equations and also for some classes of parabolic equations) is solved.

In the sixth chapter we give the conditions of existence of Bohr/Levitan almost periodic, almost automorphic Poisson stable solutions of non-autonomous perturbed gradient-like autonomous differential equations. We present here also the description of the Levinson center for gradients and gradient-likes dynamical systems with a finite number of fixed points. We establish the relation between Levinson center, chain recurrent set and Birkhoff center for compact dissipative dynamical systems.

4. LYAPUNOV STABILITY OF NON-AUTONOMOUS DYNAMICAL SYSTEMS

The foundation of the modern theory of stability was created in the works of A. Poincaré and A.M. Lyapunov.

The theory of the stability of motion has gained increasing significance in the last decade as is apparent from the large number of publications on the subject. A considerable part of these works is concerned with practical problems, especially problems from area of controls and servomechanisms, and concrete problems from engineering where the ones which first gave the decisive impetus for the expansion and modern development of stability theory.

In the last 30-40 years in the theory of stability of non-autonomous systems the substantial progress was made thanks to using the ideas and methods developed in the framework of the abstract dynamical systems (so-called "method of limiting equations").

This book contains a systematic exposition of the elements of the asymptotic stability theory of general non-autonomous dynamical systems in metric spaces with emphasis on the application for different classes of non-autonomous evolution equations (Ordinary Differential Equations (ODEs), Difference Equations (DEs), Functional-Differential Equations (FDEs), Semi-Linear Parabolic Equations etc.).

My main results about Lyapunov stability of non-autonomous dynamical systems were published in the monograph [1].

This monograph consists of four chapters.

In the first chapter we study the problem of asymptotic stability for autonomous dynamical systems. The different conditions that are equivalent to asymptotic stability are given. There are introduced and studied different kinds of dissipativity. It is established the relation between different types of attractivity. Criteria of point, compact and local dissipativity are provided. The notion of compact global attractor (Levinson center) for compact dissipative system is introduced.

The second chapter is dedicated to the asymptotic stability of non-autonomous dynamical systems. We introduce and study a special class of non-autonomous dynamical systems (non-autonomous systems with convergence). It is shown that a non-autonomous dynamical system with convergence admits a compact invariant set which is globally uniformly asymptotically stable. For general non-autonomous dynamical systems, we generalize the well-known Barbashin-Krasovskii theorem. We give a positive answer for two well-known conjectures (Markus-Yamabe and G. Sell's conjectures) for abstract non-autonomous dynamical systems and we apply the obtained general results for different classes of non-autonomous differential/difference equations (ODEs in Banach spaces, FDEs, DEs, some classes of Partial Differential Equations (PDEs)).

In the third chapter we study the problem of asymptotic stability for linear non-autonomous dynamical systems (in particular, for almost periodic, almost automorphic and recurrent systems). The bounded solutions, the relation between different types of stability and uniform exponential stability for those systems are studied. We give several applications of the obtained results for linear ODEs, PDEs, DEs and FDEs.

The fourth chapter is dedicated to the study of absolute asymptotic stability of differential/difference equations and inclusions. We establish the relation between linear inclusions and non-autonomous dynamical systems (cocycles). In the framework of general non-autonomous dynamical systems (both linear and non-linear) we study the problem of asymptotic stability and absolute asymptotic stability for discrete linear inclusions. We also study asymptotic stability of switched systems. We show that every switched system generates a non-autonomous dynamical system (cocycle). Using this fact, we apply the ideas and methods of the theory of non-autonomous dynamical systems to the study the problem of asymptotic stability of different classes of switched systems (linear systems, homogeneous systems, slow switched systems etc.).

In the last few years, I have been dealing with global attractors of almost periodic Bohr/Levitan solutions of monotone non-autonomous dynamical systems. The results obtained in this direction are reflected in a series of my publications in recent years.

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